Handout for Meeting 7

Implication-Space Semantics: The Pure Theory of Conceptual Roles

A *vocabulary* V = <L,R> is i) a lexicon L together with ii) a specification R of *reason relations* defined on that lexicon. (i) is just a set of sentences. (ii) can be a set of ordered pairs <X,Y> of sets of sentences of L, where <X,Y>∈R, for X,Y subsets of L and Y is nonempty means that X|~Y is a good (multisuccedent) implication, and if Y=∅, then <X,Y>∈R means that X is incoherent (so any subset of it is incompatible with the remainder).

An *implication* ***space*** *S* is the set of all *candidate implications* defined on a set of bearers, for instance, a lexicon L, as the set of all ordered pairs of sets of bearers: *S* = 𝒫 (L)x𝒫(L).
An *implication* ***frame*** <*S*, **I**>is an implication space together a set of distinguished candidate implications **I**⊆*S*, interpreted as the *good* implications.
Implication frames are really just encodings of vocabularies, in our technical sense.

<u>Step One: Semantically interpret candidate *implications*</u>

The bivalued *extension* of a candidate implication <X,Y> is its goodness value,
that is, whether or not <X,Y>∈**I**, meaning that it is a *good* implication, one that holds.
The *intension* of a candidate implication <X,Y> is its *range of subjunctive robustness*.

The *range of subjunctive robustness* RSR(<X,Y>) of a candidate implication is the set of pairs of sets of sentences that can be added to <X,Y> to *keep* it good (if it is in **I**) or *make* it good (if it is not in **I**):
RSR(<X,Y>) $=_{df.}$ {<W,Z>∈*S*: <X∪W,Y∪Z>∈**I**}.
Note: The RSR of a *set* of implications is the intersection of the RSRs of the elements of the set.
It follows that **<X,Y>⊆I iff <∅,∅>∈RSR(<X,Y>)**.
So implicational *intension* determines implicational *extension*, without further information.

Candidate (sets of) implications are *implicationally equivalent* iff they have the same RSRs:
G≈F iff RSR(G) = RSR(F), for G,F⊆*S*.
Implicational roles **R(G)** of (sets of) implications are their *implicational equivalence classes*:
R(G)={H⊆*S*: H≈G}.

We define two operations on implicational equivalence classes:
⊔: **adjunction** R(<X,Y>)⊔R(<W,Z>) $=_{df.}$ R(<X∪W, Y∪Z>).
⊓: **symjunction** R(<X,Y>)⊓R(<W,Z>) $=_{df.}$ R(<X,Y> ∪ <W,Z>).
for <X,Y>,<W,Z>∈*S*, and with corresponding definitions for sets of implicational ≈-classes.
As their symbols indicate, *ad*junction is an analogue in this setting of set-theoretic *union*, and *sym*junction is an analogue in this setting of set-theoretic *intersection* (see Note above re RSR).

<u>Step Two: Semantically interpret *sentences* of the lexicon in terms of the semantics of implications in which they appear as premises or conclusions</u>

If A is a sentence in the lexicon L, then $\mathbf{R^{+}(A) =_{df.} R(\langle\{A\},\emptyset\rangle)}$ is the implication equivalence class of all good implications in which A appears as a *premise*. And $\mathbf{R^{-}(A) =_{df.} R(\langle\emptyset,\{A\}\rangle)}$ is the implication equivalence class of all good implications in which A appears as a *conclusion*. The pair of these is the implicational role of sentence A, $\mathbf{R(A) =_{df.} \langle R^{+}(A), R^{-}(A)\rangle}$.

We can also form a class **C** of possible *conceptual contents*, which might or might not be the implicational roles of any sentences of L. If $F, G \subseteq S$ are *any* sets of candidate implications, $\langle R(F),R(G)\rangle$ is a conceptual content, which we can call $a \in \mathbf{C}$, so that $\mathbf{R(a)=\langle a^{+}, a^{-}\rangle = \langle R(F),R(G)\rangle}$.

<u>Interpretation Function:</u>

[.] maps sentences of the language L to some conceptual contents **C**.

The logical connective clauses that an interpretation must respect are:

If $A \in L$ is an atomic sentence, then $[A] =_{df.} \langle a^{+}, a^{-}\rangle \in \mathbf{C}$.

$[\neg A] =_{df.} \quad \langle a^{-}, a^{+}\rangle$,

$[A \rightarrow B] =_{df.} \quad \langle a^{-} \sqcap b^{+} \sqcap (a^{-} \sqcup b^{+}), a^{+} \sqcup b^{-}\rangle$,

$[A \wedge B] =_{df.} \quad \langle a^{+} \sqcup b^{+}, a^{-} \sqcap b^{-} \sqcap (a^{-} \sqcup b^{-})\rangle$,

$[A \vee B] =_{df.} \quad \langle a^{+} \sqcap b^{+} \sqcap (a^{+} \sqcup b^{+}), (a^{-} \sqcup b^{-})\rangle$.

Exercise: Show that

$[A \text{ tonk } B] = \quad \langle a^{+} \sqcap b^{+} \sqcap (a^{+} \sqcup b^{+}), a^{-} \sqcap b^{-} \sqcap (a^{-} \sqcup b^{-})\rangle$.

Theorem: This semantics in terms of implication spaces and frames is *sound and complete* for the logical vocabulary of NMMS, no matter what base vocabulary both are elaborated from.
If in the base vocabulary, every implication $\langle X,Y\rangle: X \cap Y \neq \emptyset \Rightarrow \langle X,Y\rangle \in \mathbf{I}$, then CO holds in the base and also in its logical extension by NMMS, and the logical extension is *supraclassical*.
If the converse also holds, $\langle X,Y\rangle \in \mathbf{I} \Rightarrow X \cap Y \neq \emptyset$ in the base vocabulary, the pure logic that holds in all those models is just *classical logic*.

<u>Correspondence of sequent calculus *proof* theory and implication space *model* theory</u>:

"So, the rules of NMMS are not only equivalent to the semantic clauses of truth-maker theory…, but they are also equivalent to the semantic clauses of implication space semantics.
Indeed, we can formulate this correspondence in a general way as follows.

- The *first* element in the roles defined by the semantic clauses corresponds to the *left* rule in the sequent calculus, and
  The *second* element corresponds to the *right* rule in the sequent calculus.
- The roles super-scripted with a "+" stem from sentences that occur on the *left* in a top sequent, and

The roles super-scripted with a “–” stem from sentences that occur on the *right* in a top sequent.

- An *adjunction* indicates that the adjoined roles stem from sentences in a *single* top sequent.
  And a *symjunction* indicates that the symjoined roles stem from sentences that occur in *different* top sequents.

Given that the contexts are always shared in all the sequents of any rule application, using this correspondence, **the semantic clauses above uniquely determine the sequent rules of NMMS, and the other way around**.” [*RLLR* 223]

<u>NMMS</u>:

L→:
$$\begin{array}{c} \Gamma \mid\!\sim \Delta, A \quad \Gamma, B \mid\!\sim \Delta \quad \Gamma, B \mid\!\sim \Delta, A \\ \hline\hline \Gamma, A \to B \mid\!\sim \Delta \end{array}$$

R→:
$$\begin{array}{c} \Gamma, A \mid\!\sim \Delta, B \\ \hline\hline \Gamma \mid\!\sim \Delta, A \to B \end{array}$$

L¬:
$$\begin{array}{c} \Gamma \mid\!\sim \Delta, A \\ \hline\hline \Gamma, \neg A \mid\!\sim \Delta \end{array}$$

R¬:
$$\begin{array}{c} \Gamma, A \mid\!\sim \Delta \\ \hline\hline \Gamma \mid\!\sim \Delta, \neg A \end{array}$$

L∧:
$$\begin{array}{c} \Gamma, A, B \mid\!\sim \Delta \\ \hline\hline \Gamma, A \wedge B \mid\!\sim \Delta \end{array}$$

R∧:
$$\begin{array}{c} \Gamma \mid\!\sim \Delta, A \quad \Gamma \mid\!\sim \Delta, B \quad \Gamma \mid\!\sim \Delta, A, B \\ \hline\hline \Gamma \mid\!\sim \Delta, A \wedge B \end{array}$$

L∨:
$$\begin{array}{c} \Gamma, A \mid\!\sim \Delta \quad \Gamma, B \mid\!\sim \Delta \quad \Gamma, A, B \mid\!\sim \Delta \\ \hline\hline \Gamma, A \vee B \mid\!\sim \Delta \end{array}$$

R∨:
$$\begin{array}{c} \Gamma, \mid\!\sim \Delta, A, B \\ \hline\hline \Gamma \mid\!\sim \Delta, A \vee B \end{array}$$

<u>Define implication-space frames from modalized truthmaker state spaces</u>:

“We can define an implication frame for any modalized state space, $\langle S, S^{\Diamond}, \sqsubseteq \rangle$, by letting the bearers be worldly propositions, that is, pairs of sets of states from *S*, and defining the good implications by appeal to impossible states.” [224]

“It follows from Theorem 79 that if there is a truth-maker model in which exactly a particular set of implications holds among the interpreted sentences, then there is an implication-space model in which exactly the same implications hold. The theorem ensures that for every truthmaker model, there is a parallel implication frame model such that the consequence relation defined by these models coincide.” [226]

<u>Define modalized truthmaker state spaces from implication-space models</u>:

“We can also go in the other direction. If we are given an implication-space model that codifies a particular consequence relation over a language, then we can construct a truth-maker model that codifies the same consequence relation over that language.” [227]

Let the set S of states be the set of pairs of sets of sentences of the (atomic) lexicon, $\langle X,Y \rangle$ for $X,Y \subseteq L$. Define the mereological part-whole relation among states (from which fusion of states is defined) $\sqsubseteq$ by $<X,Y>\sqsubseteq<W,Z>$ iff ($X\subseteq W$ and $Y\subseteq Z$). $<X,Y>$ is a *possible* state just in case $<X,Y>\notin\mathbf{I}$. Worldly propositions are pairs of sets of states (potential truth-makers and falsity-makers)—which are interpreted here as pairs of sets of pairs of sets of atomic sentences—satisfying Fine's Exclusivity. Consequence relations among pairs of sets of worldly propositions are defined as Hlobil does: as holding when every result of fusing any truthmaker of all of the premises with any falsity-maker of all of the conclusions is an *im*possible state. It is shown that these are exactly those determined by **I** in the original implication-space model.

<u>Premissory and Conclusory Implicational Roles in Implication-Space Semantics:</u>

Recall from last time that a *premissory* role inclusion relation $A\subseteq_P B$ holds iff A can be substituted everywhere for B as the *premise* of an implication, *salva consequentia*, and a *conclusory* role inclusion relation $A\subseteq_C B$ holds iff B can be substituted everywhere for A as the *conclusion* of an implication, *salva consequentia*.
These metainferential relations are easy to express in the implication-space setting.
For $[A]=<a^+, a^->\in\mathbf{C}$ and $[B]=<b^+, b^->\in\mathbf{C}$,
$\mathbf{A\subseteq_P B}$ **iff** $\mathbf{a^+\subseteq b^+}$ and $\mathbf{A\subseteq_C B}$ **iff** $\mathbf{b^-\subseteq a^-}$
In light of the observation last week that K3 is the logic of premissory role inclusions and LP is the logic of conclusory role inclusions, these facts indicate how to provide sound and complete semantics for those logics in the implication-space setting.

<u>Metalinguistic Functionalism about Reason Relations:</u>

Four Universal Rational Metavocabularies

A. <u>Extrinsic-Explanatory</u>:
1. Pragmatic: Bilateral, Two-sorted Deontic Normative Pragmatic Metavocabulary
2. Semantic: Truthmaker Semantics with the Hlobil Consequence Relation

B. <u>Intrinsic-Explicative</u>:
3. Logical: NMMS codes an LX logical extension of arbitrary base vocabularies, specified in the metainferential vocabulary of the multisuccedent sequent calculus.
4. Conceptual Roles: Implication-Space Semantics.

Hlobil proves an isomorphism between (1) and (2) at the level of reason relations.
(4), Implication-space semantics characterizes what is in common between (1) and (2), and shows that that is also what is expressed by the logical vocabulary of (3).
Implication-space semantics is the vocabulary of pure conceptual roles or rational forms, which are articulated by reason relations.

Metafunctionalist Claim: Reason relations are just whatever can play these specific roles with respect to *all* four of these kinds of metavocabulary.